NASA TT F-11,722

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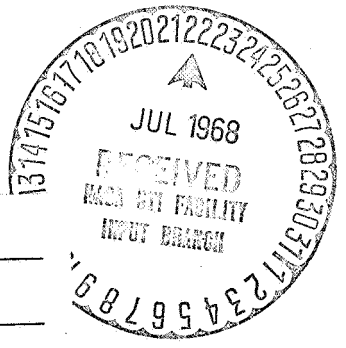
Translation of "Rasprostraneniye Vozmushcheniy v Vyazko-Uprugom i Vyazko-Plasticheskom Sterzhne"
Prikladnaya Matematika i Mekhanika, Vol. 14,
pp. 295-302, 1950



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(ACCESSION NUMBER) (THRU)
14 1
(PAGES) (CODE)
✓ 32
(NASA CR OR TMX OR AD NUMBER) (CATEGORY)
FACILITY FORM 602

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546 JUNE 1968

NASA TT F-11,722

PROPAGATION OF PERTURBATIONS IN A VISCOUS-ELASTIC AND A VISCOUS-PLASTIC ROD

I.N. Zverev

ABSTRACT. This article considers the propagation of perturbations in viscous-elastic and viscous-plastic rods. It has been shown that deformation velocity is infinite at the start of the shock despite viscosity. The deformation has been determined for colliding bodies of various masses for circular cross-section and maximal deformation has been found. The viscosity of a rod during a viscous-plastic shock can be found experimentally on the basis of this work. The article is accompanied by figures and tables, and the equations and formulas used in the calculations are also presented.

In this article we will examine the propagation of perturbations in a semi-infinite rod, the material of which displays viscous-elastic and viscous-plastic properties. It will be assumed here that stress is a linear function both of deformation and of their time derivatives; a function of a very general nature was taken from the article by A.Yu. Ishlinskiy [1]. A.N. Geracimov [2] proved the uniqueness theorem for the motion equation obtained. /295*

V.V. Sokolovskiy [3] worked on the problem of the propagation of waves in elastic-viscous-plastic rods in a somewhat different statement of the problem. The equations which V.V. Sokolovskiy used for the leading edge of the shock wave result in the relationship between stress and deformations which occur in the case of the propagation of elastic waves. The equation obtained in this experiment does not exhibit this restriction, although it does not consider completely accurately all processes which occur during the shock.

In the case of a collision of a body with a viscous-elastic semi-infinite rod, the solution has been calculated in §§1 - 5 and it has been shown that the deformation velocity is infinite at the beginning of the shock, in spite of viscosity. The deformation for circular cross-section has been computed for

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* Numbers in the margin indicate pagination in the foreign text.

colliding bodies of various masses and the maximal deformation has been found. The latter part of this work (§ 6) makes it possible to determine experimentally the viscosity of a rod during a viscous-plastic shock.

§ 1. Let us consider the problem of plane perturbations through a semi-infinite rod, on the end of which is given the velocity $v(0, t)$ or deformation $\varepsilon(0, t)$. We will denote the axial stress in cross-section x by $\sigma(x, t)$ and axial deformation by $\varepsilon(x, t)$. We will denote the displacement of cross-section x by $u(x, t)$ and the density by ρ.

The motion equation, as we know, has the form:

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x} \tag{1.1}$$

In the case of a viscous-elastic body the stress components are functions of the deformation decomponents or of their time derivatives. For the one-dimensional problem we obtain

$$\sigma(x, t) = E\varepsilon + \mu \frac{\partial \varepsilon}{\partial t} \tag{1.2}$$

where μ is a coefficient of viscosity and E is the modulus of elasticity.

By substituting σ in the motion equation we obtain

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^3 u}{\partial x^2 \partial t} \qquad \left(a^2 = \frac{E}{\rho},\ \nu = \frac{\mu}{\rho}\right) \tag{1.3}$$

The boundary and initial conditions will be

$$v = u_t(0, t) = \varphi(t) \tag{1.4}$$

$$\varepsilon(\infty, t) = u_x(\infty, t) \tag{1.5}$$

$$u(x, 0) = 0, \qquad [\partial u(x, t)/\partial t]_{t=0} = 0 \tag{1.6}$$

where $\phi(t)$ is a given function and $u_x(\infty, t)$ is a bounded value.

§ 2. We will solve the motion equation by the operational method [4]. /296

For this purpose we will multiply equation (1.2) by $e^{-pt}dt$ and will integrate it from zero to infinity, and we will then multiply the expression obtained by p. By designating

$$u(x, t) \doteqdot U(x, p) = p\int_0^\infty e^{-pt} u(x, t)\, dt \tag{2.1}$$

and recalling the zero initial data (1.6) we will obtain the motion equation in operational form

$$p^2U(x, p) = (a^2 + \nu p)\frac{\partial^2 U(x, p)}{\partial x^2} \tag{2.2}$$

The solution of (2.2) which satisfies condition (1.5) at infinity will be

$$U(x, p) = A(p)\exp\frac{-xp}{\sqrt{a^2 + \nu p}} \tag{2.3}$$

By using the boundary condition (1.4) for determining A(p) we have from (2.3):

$$A(p) = \frac{\Phi(p)}{p}, \quad \varphi(t) \doteqdot \Phi(p)$$

Let

$$\psi(x, t) \doteqdot \exp\frac{-xp}{\sqrt{a^2 + \nu p}}$$

Then, in accordance with the multiplication theorem, the solution of the boundary value problem may be written in the form

$$u(x, t) = \int_0^t \varphi(\tau)\psi(t - \tau, x)\, d\tau \tag{2.4}$$

We will now find the initial function $\psi(x, t)$ in accordance with the transformation:

$$\psi(x, t) \doteqdot \exp\frac{-xp}{\sqrt{a^2 + \nu p}} = \exp\left\{-x_1\sqrt{p + \alpha} + \frac{x_1\alpha}{\sqrt{p + \alpha}}\right\} \tag{2.5}$$

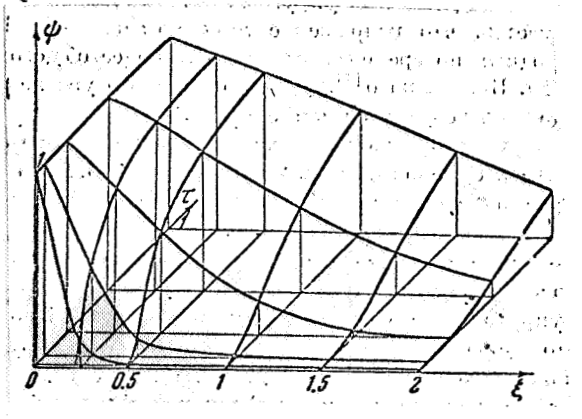

Fig. 1.

Here $\alpha = a^2/\nu$, $x_1 = x/\sqrt{\nu}$. By introducing yet another designation $x_1\sqrt{\alpha} = xa/\nu = \xi$, where ξ is a dimensionless value, and by changing over to the new complex variable $q = p/\alpha$, we will find according to the displacement theorem

$$\psi(x, t/\alpha) = \frac{q}{q - 1}\exp\left\{-\xi\sqrt{q} + \frac{\xi}{\sqrt{q}}\right\} \tag{2.6}$$

Now let us examine the initial functions

$$\varphi_1(t) \doteqdot \exp\left\{-\xi\sqrt{q} + \frac{\xi}{\sqrt{q}}\right\},$$

$$\varphi_2(t) \doteqdot \frac{q}{q-1} = e^t$$

We will examine beforehand the auxiliary function

$$\varphi_1^*(t) \doteqdot \exp\left\{-\xi r + \frac{\xi}{r}\right\}$$

We know from the operational computation that

$$e^{\xi/r} \doteqdot I_0(2\sqrt{\xi t})$$

where I_0 (z) is Bessel's function from the imaginary argument of the zero power.

By using the delay theorem we find

$$\varphi_1^*(t) = \begin{cases} 0 & \text{for } t < \xi \\ I_0(2\sqrt{\xi[t-\xi]}) & \text{for } t > \xi \end{cases}$$

By making use of the Efros transformation we change from the function $\phi_1^*(x)$ to $\phi_1(x)$. We then find /297

$$\varphi_1(t) = \int_0^\infty \frac{\varphi^*(s)}{\sqrt{\pi t}} \exp\frac{-s^2}{4t}\, ds = \int_\xi^\infty \frac{I_0(2\sqrt{\xi[s-\xi]})}{\sqrt{\pi t}} \exp\frac{-s^2}{4t}\, ds \qquad (2.7)$$

Turning then to $\psi(x, t)$ we find by using the Borel theorem:

$$\psi(x, t/a)\, e^t \doteqdot \int_0^t \varphi_1(z)\, \varphi_2'(t-z)\, dz + \varphi_2(0)\, \varphi_1(t)$$

Hence

$$\psi(x, t) = e^{-\alpha t} \int_\xi^\infty \frac{I_0(2\sqrt{\xi[s-\xi]})}{\sqrt{\pi\alpha t}} \exp\frac{-s^2}{4\alpha t}\, ds + \int_0^{\alpha t} dz \int_\xi^\infty \frac{I_0(2\sqrt{\xi[s-\xi]})}{\sqrt{\pi z}} \exp\left(\frac{-s^2}{4z} - z\right) ds$$

We will transform the second integral in this expression. By changing the order of integration, which is possible here, and by introducing the dimensionaless variable $\tau = \alpha t$, we obtain

$$\psi(x, t) = \int_{\xi}^{\infty} I_0(2\sqrt{\xi[s-\xi]}) \left[\frac{1}{\sqrt{\pi\tau}} \exp\left(\frac{-s^2}{4\tau} - \tau\right) + \right.$$
$$\left. + \frac{1}{2}\left\{ e^{-s} \operatorname{erfc}\left(\frac{s}{2\sqrt{\tau}} - \sqrt{\tau}\right) - e^{s} \operatorname{erfc}\left(\frac{s}{2\sqrt{\tau}} + \sqrt{\tau}\right)\right\}\right] ds \qquad (2.8)$$

It is easy to find directly from (2.8), as well as from the operational representation of $\psi(x, t)$ that $\psi(0, t) = 1$, $\psi(x, \infty) = 1$ and $\psi(x, 0) = 0$.

Certain values of $\psi(\xi, \tau)$ are listed in Table 1; Figure 1 illustrates the graph constructed according to these data.

TABLE 1

τ \ ξ	0	0.25	0.5	1	1.5	2
0	1	0	0	0	0	0
0.01	1	0.102	0.001	0	0	0
0.0625	1	0.488	0.080	0.020	0	0
0.25	1	0.762	0.525	0.182	0.039	0.005
0.5625	1	0.880	0.733	0.441	0.214	0.084
1	1	0.924	0.853	0.648	0.435	0.258

§ 3. Now let us consider the elastic-viscous longitudinal shock in a semi-infinite rod with an absolutely solid body and mass m. We will look for deformation $\varepsilon(x, t)$. By differentiating (2.3) with respect to x, we obtain

$$-u_x(x, t) \doteq \frac{A(p)\, p}{\sqrt{a_0^2 + \nu p}} \exp\left\{-\frac{xp}{\sqrt{a_0^2 + \nu p}}\right\} \qquad (3.1)$$

In this case boundary condition (1.4) will be

$$m\frac{\partial^2 u}{\partial t^2} = F_0\sigma \quad \text{for } x = 0 \qquad (3.2)$$

Here the stress for the elastic-viscous rod is found by formula (1.2).

From (3.2) and (1.2) we will have

$$-pv_0 - p^2 u(0, 0) + p^2 A = \{(E + \mu p) U_x(0, p) - \mu p \varepsilon(0, 0)\} F_0/m$$

where v_0 is the initial velocity of the shock. By substituting $U_x(0, p)$ here /298 in accordance with (3.1), solving with respect to A, and recalling that $\varepsilon(0, 0) = 0$, we obtain

$$A = \frac{v_0}{p + b\sqrt{a^2 + \nu p}} \qquad \left(b = \frac{F_0 \rho \sqrt{\nu}}{m}\right) \tag{3.3}$$

By substituting expression (3.3) into (3.2) we find

$$\varepsilon(x, t) \doteqdot \frac{v_1 p}{\sqrt{p+\alpha}\,(p + b\sqrt{p+\alpha})} \exp \frac{-x_1 p}{\sqrt{p+\alpha}} \qquad \left(v_1 = \frac{v_0}{\sqrt{\nu}}\right) \tag{3.4}$$

We will then find the initial function from the representation of (3.4). From the displacement theorem we find

$$-\varepsilon(x, t)\, e^{\alpha t} = \frac{v_1 \sqrt{p}}{p + b\sqrt{p-\alpha}} \exp\left\{-x_1\sqrt{p} + \frac{x_1 \alpha}{\sqrt{p}}\right\} \tag{3.5}$$

The initial function of the second factor is known. The first factor is represented in the form

$$f_1(t) = \frac{v_1}{p_1 - p_2}\left[\frac{p\sqrt{p}}{p-p_1} - \frac{p\sqrt{p}}{p-p_2} - \frac{\alpha\sqrt{p}}{p-p_1} + \frac{\alpha\sqrt{p}}{p-p_2} - \frac{bp}{p-p_1} + \frac{bp}{p-p_2}\right] \tag{3.6}$$

where

$$p_{1,2} = \tfrac{1}{2}(2\alpha + b^2 \pm b\sqrt{4\alpha + b^2})$$

We know from operational computation that

$$\frac{p}{p-p_i} \doteqdot e^{p_i t}, \quad \frac{p\sqrt{p}}{p-p_i} \doteqdot \sqrt{p_i} e^{p_i t} \operatorname{erf}\sqrt{p_i t} + \frac{1}{\sqrt{\pi t}}, \quad \frac{\sqrt{p}}{p-p_i} \doteqdot \frac{1}{\sqrt{p_i}} e^{p_i t} \operatorname{erf}\sqrt{p_i t}$$

By substituting these expressions in (3.5) we find

$$f_1(t) = \frac{v_1}{p_1 - p_2}\left\{e^{p_1 t}\left[\left(\sqrt{p_1} - \frac{\alpha}{\sqrt{p_1}}\right)\operatorname{erf}\sqrt{p_1 t} - b\right] - \right.$$
$$\left. - e^{p_2 t}\left[\left(\sqrt{p_2} - \frac{\alpha}{\sqrt{p_2}}\right)\operatorname{erf}\sqrt{p_2 t} - b\right]\right\}$$

Since p_1 and p_2 are the roots of the quadratic equation $p^2 - (2\alpha + b)\, p + \alpha^2 = 0$, it is not difficult to prove:

$$\sqrt{p_1} - \frac{\alpha}{\sqrt{p_1}} = b, \quad \sqrt{p_2} - \frac{\alpha}{\sqrt{p_2}} = -b$$

Therefore

$$f_1(t) = \frac{v_1 b}{p_1 - p_2}\left[2e^{p_2 t} - e^{p_2 t}\operatorname{erfc}\sqrt{p_2 t} - e^{p_1 t}\operatorname{erfc}\sqrt{p_1 t}\right], \quad f_1(0) = 0$$

Now, by applying Borel's theorem to (3.5) and turning to the dimensionless variables $\tau = \alpha t$ and $\xi = xa/\nu$ we obtain

$$-\frac{\varepsilon}{\varepsilon_0} = \frac{e^{-\tau}}{\sqrt{1 + b^2/4\alpha}}\int_0^\tau \left\{\left[\frac{p_2}{\alpha}\exp\frac{p_2 z}{\alpha} - \right.\right. \qquad \left(\varepsilon_0 = \frac{v_0}{a_0}\right)$$

$$- \frac{p_2}{2\alpha}\exp\frac{p_2 z}{\alpha}\operatorname{erfc}\sqrt{\frac{p_2 z}{a}} - \frac{p_1}{2\alpha}\exp\frac{p_1 z}{\alpha}\operatorname{erfc}\sqrt{\frac{p_1 z}{\alpha}} + \tag{3.7}$$

$$\left. + \frac{\sqrt{p_1} + \sqrt{p_2}}{2\sqrt{\pi\alpha z}}\right]\int_\xi^\infty I_0\left(2\sqrt{\xi[s-\xi]}\right)\exp\frac{-s^2}{4(\tau - z)}\,\frac{ds}{\sqrt{\pi(\tau - z)}}\Bigg\}\,dz$$

§ 4. Let us see now how deformation ε of circular cross-section (x = 0) changes during the time of the shock. We will note that the internal integral in formula (3.7) is equivalent in this case to unity. By introducing the designation /299

$$\lambda = \frac{\rho\mu^2 F_0^2}{2Em^2}, \qquad k = 1 + \lambda - \sqrt{\lambda^2 + 2\lambda} \tag{4.1}$$

where λ is a dimensionless value, we have

$$-\frac{\varepsilon(0, \tau/\alpha)}{\varepsilon_0} = \frac{1}{\sqrt{1 + \lambda/2}}\left[e^{(k-1)\tau} - \frac{e^{(k-1)\tau}\operatorname{erfc}\sqrt{k\tau}}{2} - \frac{e^{(1/k-1)\tau}\operatorname{erfc}\sqrt{\tau/k}}{2}\right]$$

1. In the case of the shock of a body of infinite mass ($m \to \infty$) we find

$$-\frac{\varepsilon(0, \tau/\alpha)}{\varepsilon_0} = \operatorname{erf}\sqrt{\tau}, \quad \text{or} \quad -\varepsilon(0, t) = \frac{v_0}{a_0}\operatorname{erf}\sqrt{\frac{a_0^2 t}{\nu}} \tag{4.3}$$

From here we see that in the case of the shock of infinite mass through a viscous-elastic semi-infinite rod the deformation of circular cross-section from zero increases rapidly (but not as much as in the elastic case) from zero to elastic, equal to v_0/a_0.

2. In the case of an elastic shock (viscosity $\nu = 0$) we find from (4.1) equation

$$-\varepsilon(0, \tau/\alpha) = \frac{v_0}{a_0} \tag{4.4}$$

3. In the case of the shock of a body of finite mass, $\varepsilon(0, \infty) = 0$ in accordance with (4.2), since $k < 1$.

Figure 2 illustrates the curves of (4.2) for colliding bodies of various masses. The reaction time of the blow, as we can see, is short for large values of λ (when the colliding body has a small mass) and transverse compression cannot be disregarded.

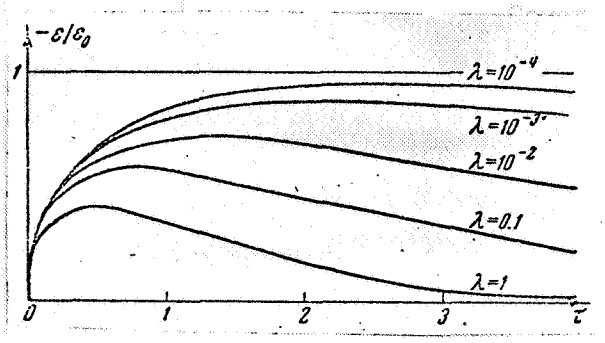

Fig. 2.

Figure 2 shows also that when $t = 0$ the deformation velocity $\partial\varepsilon/\partial t$ will be maximum. By arranging the expression for $\partial\varepsilon/\partial t$ in accordance with (4.2) it is easy to establish that in the case of the shock of a body through a viscous-elastic semi-infinite rod the velocity of deformation will be infinite for $t = 0$.

The second and third terms of the left half of formula (4.2) differ little in total from $\operatorname{erfc}\sqrt{\tau}$ for small values of λ. Since λ must be small during a longitudinal shock, we may make use of the approximate formula

$$\frac{\varepsilon}{\varepsilon_0} = \frac{1}{\sqrt{1+\lambda/2}}\left[\exp(k-1)\tau - \operatorname{erfc}\sqrt{\tau}\right] \qquad (4.5)$$

We now find ε_{max}. By differentiating (4.5) with respect to t and equating to zero we will find

$$\frac{\pi(k-1)^2}{2k}\, y = e^{-y} \qquad (y = 2k\tau_{max})$$

Hence for each value of λ we will find τ_{max}; by substituting τ_{max} in (4.5) we determine ε_{max}. As we see from Figure 3, the maximal deformation is a function of λ and trends toward elastic deformation when $\lambda \to 0$.

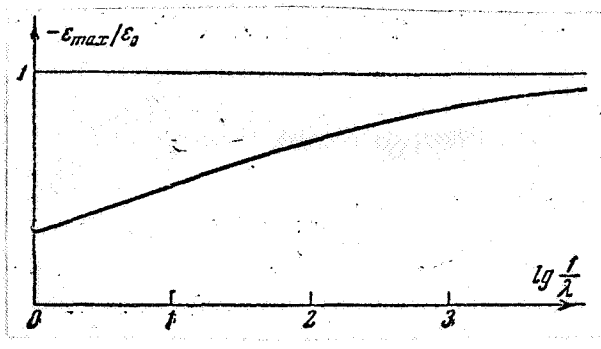

Fig. 3.

§ 5. Let us take a look at the solution for small values of λ. By placing λ in (3.7) in accordance with (4.1) and making use of the smallness of λ we will find /300

$$-\frac{\varepsilon}{\varepsilon_0} = \frac{e^{-\tau}}{\sqrt{1+\lambda/2}}\int_0^{\tau}\left[ke^{kz} - e^{z}\operatorname{erfc}\sqrt{z} + \right.$$

$$\frac{1}{\sqrt{\pi z}}\Big]\left\{\int_{\xi}^{\infty} I_0\left(2\sqrt{\xi[s-\xi]}\right)\exp\frac{-s^2}{4(\tau-s)}\,\frac{ds}{\sqrt{\pi(\tau-z)}}\right\}dz \tag{5.1}$$

Table 2 lists the results of calculations of $\varepsilon/\varepsilon_0$ for $\lambda = 10^{-2}$ and Figure 4 illustrates the graphs of the value of $\varepsilon/\varepsilon_0$ as a function of τ and ξ constructed on the basis of these data.

TABLE 2

ξ \ τ	1	1.5	2	2.5	3	3.5	4	4.5	5
0	0.720	0.740	0.725	0.690	0.660	0.625	0.590	0.560	0.525
1	0.396	0.528	0.610	0.646	0.661	0.650	0.635	—	—
2	0.127	0.268	0.392	0.498	0.552	0.603	0.607	0.609	0.597

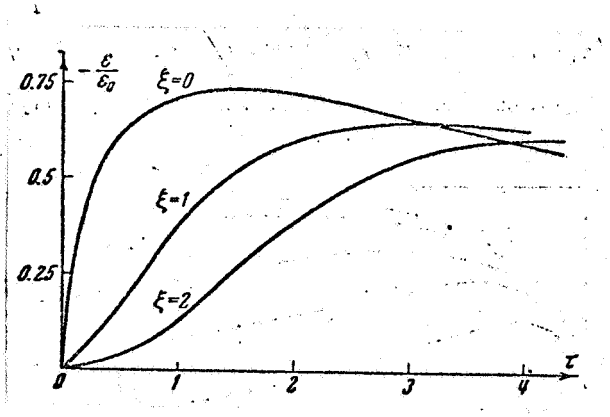

Fig. 4.

The propagation velocity of perturbation which corresponds to the solution of (3.7) and (5.1) is infinite while the velocity of the motion of the maximum of the perturbation is finite, whereupon the maximum drops by measure of increasing x. Let us look for comparison at the shock through a completely viscous semi-infinite rod. The equation of the motion will be

$$\frac{\partial^2 u}{\partial t^2} = \nu\frac{\partial^3 u}{\partial x^2 \partial t} \tag{5.2}$$

By solving this equation by the operational methods we find the following expression

$$-\varepsilon = \frac{v_1}{b}\left[\operatorname{erfc}\frac{x}{2\sqrt{\nu t}} - \exp\left(\frac{xb}{\sqrt{\nu}} + b^2 t\right)\operatorname{erfc}\left(\frac{x}{2\sqrt{\nu t}} + b\sqrt{t}\right)\right] \tag{5.3}$$

The deformation $-\varepsilon$, as we can easily see, is an increasing function, where $-\varepsilon = v_0 m/F_0\mu$ for $t \to \infty$. With the shock of an infinite mass the deformation increases without bounds with the passage of time.

§ 6. Let us now examine the shock of an absolutely solid body on a semi-infinite viscous-plastic rod which has been preliminarily compressed to the limits of elasticity. The initial conditions will be

$$u_x(x, 0) = \varepsilon_s, \qquad u_t(x, 0) = 0 \tag{6.1}$$

The boundary conditions are

$$u_x(\infty, t) = \varepsilon(\infty, t), \qquad mu_{tt}(0, t) = F_0\sigma \tag{6.2}$$

where $\varepsilon(\infty, t)$ is a finite value and

$$\sigma = \sigma_s + E'(\varepsilon - \varepsilon_s) + \mu\frac{\partial\varepsilon}{\partial t} \tag{6.3}$$

Here E' is the modulus of hardening.

We will now introduce a new variable $\varepsilon^* = \varepsilon - \varepsilon_s$. Then the initial conditions will be /301

$$u_x^*(x, 0) = 0, \qquad u_t^*(x, 0) = 0 \tag{6.4}$$

and expression (6.3) for σ will become:

$$\sigma = \sigma_s + E'\varepsilon^* + \mu\varepsilon_t^* \tag{6.5}$$

The solution of equation (1.3) in this case will be

$$u^*(x, t) \doteqdot A\exp\frac{-xp}{\sqrt{a_1^2 + \nu p}} \qquad \left(a_1^2 = \frac{E'}{\rho}\right) \tag{6.6}$$

We will find A from the second boundary condition (6.2). We have

$$-pv_0 + p^2A = \frac{F_0}{m}(E' + p\mu)\frac{-Ap}{\sqrt{a_1^2 + \nu p}} + \frac{F_0}{m}\sigma_s$$

Here we make use of the condition $\varepsilon^*(0, 0) = 0$. Hence we have

$$A = A_1 + A_2$$

where

$$A_1 = \frac{v_0}{p + b\sqrt{p + \alpha_1}}, \qquad A_2 = \frac{w_0}{p(p + b\sqrt{p + \alpha_1})}, \qquad w_0 = \frac{F_0}{m}\sigma_s, \qquad \alpha_1 = \frac{a_1^2}{\nu}$$

Therefore the solution of (6.6) becomes:

$$u^* \doteq A_1 \exp \frac{-xp}{\sqrt{a_1^2+\nu p}} + A_2 \exp \frac{-xp}{\sqrt{a_1^2+\nu p}}$$

For deformation we obtain the expression

$$-\varepsilon^*(x, t) = \varepsilon_1^*(x, t) + \varepsilon_2^*(x, t) \qquad \left(\varepsilon_{1,2}^* = \frac{A_{1,2}}{\sqrt{a_1^2+\nu p}} \exp \frac{-xp}{\sqrt{a_1^2+\nu p}}\right)$$

The initial function of the first term of $\varepsilon_1^*(x, t)$ has been computed. Let us find the initial function of the second term. From the displacement theorem we have

$$\varepsilon_2^* e^{\alpha t} = \left[\frac{w_1\sqrt{p}}{(p-p_1)(p-p_2)} - \frac{w_1 bp}{(p-p_1)(p-p_2)(p-\alpha_1)}\right] \exp\left(\frac{x_1\alpha_1}{\sqrt{p}} - x_1\sqrt{p}\right)$$

where

$$p_{1,2} = \frac{2\alpha_1 + b^2 \pm \sqrt{4\alpha_1 + b^2}}{2} \qquad \left(w_1 = \frac{w_0}{\sqrt{\nu}} = \frac{F\sigma_s}{m\sqrt{\nu}}\right)$$

We similarly obtain

$$\varphi_1(t) \doteq \frac{w_1\sqrt{p}}{(p-p_1)(p-p_2)} \doteq \frac{w_1}{p_1-p_2}\left[\frac{e^{p_1 t}}{\sqrt{p_1}} \operatorname{erf}\sqrt{p_1 t} - \frac{e^{p_2 t}}{\sqrt{p_2}} \operatorname{erfc}\sqrt{p_2 t}\right]$$

$$\varphi_2(t) \doteq \frac{w_1 bp}{(p-p_1)(p-p_2)(p-\alpha_1)} \doteq \frac{w_1 b}{p_1-p_2}\left[\frac{e^{p_1 t}}{p_1-\alpha_1} - \right.$$

$$\left. - \frac{e^{p_2 t}}{p_2-\alpha_1} + \frac{(p_1-p_2)e^{\alpha_1 t}}{(p_1-\alpha_1)(p_2-\alpha_1)}\right]$$

$$f_2(t) \doteq \exp\left(\frac{x_1\alpha_1}{\sqrt{p}} - x_1\sqrt{p}\right) \doteq \int_{\xi}^{\infty} I_0(2\sqrt{\xi[s-\xi]}) \exp\frac{-s^2}{4\alpha_1 t} \frac{ds}{\sqrt{\pi\alpha_1 t}}$$

By designating $\phi_1(t) - \phi_2(t) = \phi_{12}(t)$, we have from Borel's formula

$$\varepsilon_2^* = \int_0^t \varphi_{12}'(z) f_2(t-z)\, dz$$

For the deformation of circular cross-section we find /302

$$-\varepsilon^*(0, t) = \frac{v_1 b}{p_1-p_2}\left[2e^{(p_2-\alpha_1)t} - e^{(p_2-\alpha_1)t}\operatorname{erfc}\sqrt{p_2 t} - e^{(p_1-\alpha_1)t}\operatorname{erfc}\sqrt{p_1 t}\right] +$$

$$+ \frac{w_1 b}{p_1-p_2}\left[\frac{2}{p_2-\alpha_1}e^{(p_2-\alpha_1)t} - \frac{\operatorname{erfc}\sqrt{p_2 t}}{p_2-\alpha_1}e^{(p_2-\alpha_1)t} - \frac{\operatorname{erfc}\sqrt{p_1 t}}{p_1-\alpha_1}e^{(p_1-\alpha_1)t}\right] + \frac{p_1-p_2}{\alpha b^2} \qquad (6.7)$$

Hence $\varepsilon^*(0, 0) = 0$ and $\varepsilon^*(0, \infty) = -\sigma_s/E$.

1. For a plastic shock (viscosity $\nu = 0$) we have

$$p_1 - \alpha_1 = \frac{F_0 \rho a_1}{m}, \qquad p_2 - \alpha_1 = -\frac{F_0 \rho a_1}{m}$$

Consequently,

$$-\varepsilon^*(0, t) = \left(\frac{v_0}{a_1} - \frac{\sigma_s}{E'}\right) \exp \frac{-F_0 \rho a_1 t}{m} + \frac{\sigma_s}{E'}$$

Hence $\varepsilon^*(0, 0) = -v_0/a_1$ and $\varepsilon^*(0, \infty) = -\sigma_s/E'$.

2. In the case of the shock of a body of infinite mass ($m \to \infty$) we have

$$\begin{array}{lll} p_1 = \alpha_1, & m(p_1 - \alpha_1) = & F_0 \rho a_1, \\ p_2 = \alpha_2, & m(p_2 - \alpha_1) = & -F_0 \rho a_1, \end{array} \qquad \frac{p_1 - p_2}{b} = 2\frac{a_1}{\sqrt{\nu}}$$

Consequently,

$$-\varepsilon^*(0, t) = \frac{v_0}{a_1} \operatorname{erf} \sqrt{\alpha_1 t}$$

By calculating the maximal deformation of circular cross-section $x = 0$ for $E = 2 \times 10^6$ kg/cm^{-2}, $E' = 0.04\ E$, $\sigma_s = 2000$ kg/cm^{-2}, $v_0 = 25$ m/sec^{-1},

$$\frac{F_0 \rho \mu^2}{2E' m^2} = 10^{-2}$$

we then obtain

$$-\varepsilon_{max}^* \frac{a_1}{v_0} = 0.74 + 0.8 \times 10^{-4}$$

It is clear that the second term in formula (6.7) can be disregarded when $x = 0$.

REFERENCES

1. Ishlinskiy, A. Yu., *PMM*, Vol. 4, No. 1, 1940.
2. Gerasimov, A. N., "The Problem of Elastic Recoil and Internal Friction," *PMM, New Series*, Vol. 1, No. 4, 1938.
3. Sokolovskiy, V. V., "Propagation of Elastic-Fluid-Plastic Waves in Rods," *PMM*, Vol. 12, No. 3, 1948.
4. Efros, A. M. and A. M. Danilevskiy, *Operatsionnoye Ischisleniye i Konturnyye Integraly* [Operational Calculus and Contour Integrals], Kharkov, 1937.

Translated for the National Aeronautics and Space Administration under contract No. NASw-1695 by Techtran Corporation, P.O. Box 729, Glen Burnie, Maryland 21061